Elements of Probability and Statistics

Lecture 01: Probability and Counting

IAI, TCG-CREST August 08, 2023

Textbook for Probability: *Introduction to Probability (2nd ed.), by Joseph K. Blitzstein and Jessica Hwang, CRC Press, 2019.*

1.1 Introduction

In this course of the "Elements of Probability and Statistics", the entities under our focus of study are stochastic mathematical systems.

A mathematical system can be informally described as a mathematical structure involving one or more sets of objects with rules that define relations between them.

A mathematical system is said to be deterministic if it has no randomness or uncertainty in its behavior, i.e., it always produces the same output or result from a given input or initial state. Some examples of deterministic mathematical systems:

- If a ball is thrown up in the air with a known amount of force, we can calculate the exact height up to which it travels.
- If we know the length of a simple pendulum, we can calculate the exact time period of its motion.
- From the measurement of the radius of a circle, its exact circumference can be calculated.

There are however, systems whose behaviour cannot be predicted in such a definite manner. Such system are called probabilistic or stochastic systems. Some examples of stochastic systems are:

- From the roll of an impartial six-sided dice, the number that will come up on the upper face of the dice when it comes to rest cannot be predicted.
- Near-future weather conditions such as temperature, humidity, wind speed, etc. cannot be predicted in a definite manner.
- The procedure by which common human behaviours occur, such as recognizing a person from their face or voice, cannot be described in a definite deterministic manner.

Why do stochastic systems exist? The systems may involve variables that are inherently random. Alternatively, the system might actually be deterministic over a set of variables, but either all variables are currently not measurable, or the number of variables is so large that it becomes infeasible to compute the behaviour of the system.

If a system exhibits a random nature, in what manner can we study them to improve our understanding of the system? This question leads us to ask, are there constants in a stochastic system, that we can assume or estimate, that allows us to make reliable predictions on the behaviour of a stochastic system?

This leads us to the study of two closely related fields of Mathematics: probability, and statistics.

- In the study of probability, we first make assumptions on the nature of the stochastic system, for example we may assume a function that defines how likely are the different possible outcomes of the system. We then study what random outcomes will occur from this system.

- Conversely, in the study of statistics we observe random outcomes and make assumptions on the nature of a stochastic system that may have generated the observed outcomes. We then try to identify a best stochastic system which could have been responsible for the observed outcomes.

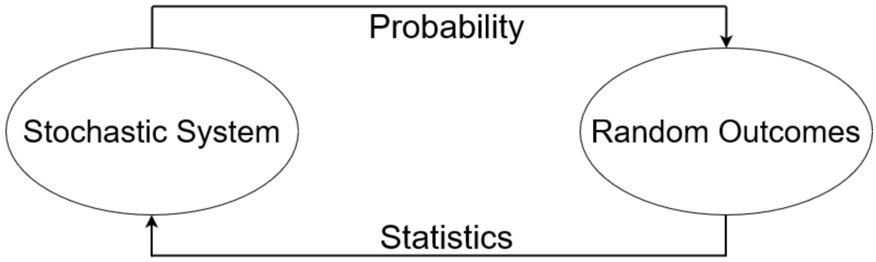

Figure 1: Relationship between the subjects of probability and statistics. Image motivated from All of Statistics, by Larry Wasserman.

1.2 A naive definition of probability

We begin with the study of probability, which serves as the foundation on which we will later study statistics. We begin with the formal specification of a framework in which we will discuss probability. The elements of this framework are as follows.

1. An outcome: A unit of random data from a stochastic system.
2. Sample space: The set of all possible outcomes from a stochastic system will be called a Sample Space S. From this definition, any outcome $x \in S$.
3. An event: A set of possible outcomes will be called an event A. Any event $A \subseteq S$.
4. A Random Experiment: In a random experiment, we obtain a single outcome x from a stochastic system. If $x \in A$, we state that event A has occurred.

The field of set theory allows us to formally state and discuss the relation between different events. Some examples of ways of stating events:

- Either event A or event B: $A \cup B$.
- Events A and B: $A \cap B$.
- not A: A^c.
- A or B, but not both: $(A \cap B^c) \cup (A^c \cap B)$.
- at least one of the events $A_1, ..., A_n$: $A_1 \cup ... \cup A_n$.
- all events $A_1, ..., A_n$: $A_1 \cap ... \cap A_n$.
- Event A always implies event B: $A \subseteq B$.
- Events A and B are mutually exclusive: $A \cap B = \phi$.
- $A_1, ..., A_n$ are a partition of S: $A_1 \cup ... \cup A_n = S$, $\forall i \neq j : A_i \cap A_j = \phi$.

The naive definition of Probability: Let A be an event from a random experiment with finite sample space S. The naive probability of A is,

$$P_{naive}(A) = \frac{|A|}{|S|}.$$

The naive definition is simplistic and is applicable only when all outcomes of a stochastic system can be assumed to be equally likely.

1.3 A review of elemental counting

- Multiplication Rule: If random experiment R_A has a number of outcomes, and random experiment R_B has b number of outcomes, then the compound experiment of R_A followed by R_B has ab number of possible outcomes.

  Q: What are the total number of outcomes, for the compound experiment of a dice roll followed by a coin toss?

  Q: There are two types of waffle cones available: Plain and Chocolate. There are two flavours of ice-cream that are available: Vanilla, Chocolate, and Strawberry. What are the total number of unique cone and ice-cream combinations that you can get?

  Q : What are the total number of subsets of a set with n elements?

- Permutations and Binomial Coefficients: A permutation of n elements $1, ..., n$ is a unique arrangement of the elements. What are the total number of permutations that are possible for n elements?

  For a set with n elements, the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ gives the number of subsets of size k.

- On sampling with or without replacement, and whether elements can or cannot be distinguished.

  Q: There is a box filled with n uniquely numbered cards. In how many ways can k cards be drawn from the box, with replacement?

  Q: For the same problem, in how many ways can k cards be drawn, without replacement?

  Q: What are the number of ways in which 3 people can be selected from a pool of n job applicants?

  Q: Bose-Einstein encoding: In how many ways can k indistinguishable particles be put into n distinguishable boxes?

	Order matters	Order does not matter
SWR	n^k	$\binom{n+k-1}{k}$
SWOR	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Table 1: Counting over four cases: Sampling with replacement (SWR), sampling without replacement (SWOR), and for both situations considering whether the order of the objects in a sample matters or not.

Q: If two fair dice are rolled, is a sum of 11 more likely, or a sum of 12?